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NOTE ON THE RANK OF A SYMMETRICAL MATRIX. II.

BY J. H. M. WEDDERBURN.

In an earlier paper with the same title, I gave a proof of the theorem,* that in a symmetrical matrix of rank r there is a principal minor of order r which does not vanish, by first showing that the sum of the principal minors of order r is not zero. This is however not necessarily true when the coefficients of the matrix are complex as is seen from the example†

$$\begin{vmatrix} 1 & i \\ i & -1 \end{vmatrix}.$$

The origin of this exceptional case lies in the fact that it is not possible to find an orthogonal matrix which will transform the point, or vector, $(x_{11}, x_{12}, \dots x_{1n})$ into $(1, 0, \dots 0)$ when

$$(1) \quad \sum_{s=1}^n x_{1s}^2 = 0.$$

It is however easily shown by induction that the required orthogonal matrix does exist when (1) is not satisfied. In extending the theorem to complex coefficients we have therefore only to consider the effect of null vectors of the given matrix whose tensors are zero.

If the tensor of‡

$$x + iy = (\xi_1 + i\eta_1, \xi_2 + i\eta_2, \dots \xi_n + i\eta_n)$$

is zero, we have

$$\sum_{s=1}^n \xi_s^2 = \sum_{s=1}^n \eta_s^2, \quad \sum_{s=1}^n \xi_s \eta_s = 0.$$

The two vectors, $x = (\xi_1, \xi_2, \dots \xi_n)$ and $y = (\eta_1, \eta_2, \dots \eta_n)$, are therefore orthogonal and, seeing that their tensors are equal and not zero, there exists an orthogonal matrix which transforms them into e_1 and e_2 respectively, e_s denoting the vector all of whose coefficients are zero except the s th which is 1.

Suppose now that $x + iy$ is a null vector of a matrix, A , whose rank is r . If A is transformed by the orthogonal matrix determined above,

* Frobenius, "Über das Pfaffsche Problem," Crelle, vol. 82, 1877, p. 242.

† This example is due to Frobenius.

‡ By the sum, $x + y$, of two vectors is meant the vector obtained by adding corresponding coefficients of x and y .

the transformed matrix has $e_1 + ie_2$ as a null vector and therefore has the form

$$A_1 \equiv \begin{vmatrix} \alpha & i\alpha & \beta_3 & \beta_4 & \cdots & \beta_n \\ i\alpha & -\alpha & i\beta_3 & i\beta_4 & \cdots & i\beta_n \\ \beta_3 & i\beta_3 & a_{33} & a_{34} & \cdots & a_{3n} \\ \beta_4 & i\beta_4 & a_{34} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \beta_n & i\beta_n & a_{3n} & \cdot & \cdot & \cdot \end{vmatrix}.$$

A_1 has the same rank as the matrix

$$A_2 \equiv \begin{vmatrix} \alpha & \beta_3 & \beta_4 & \cdots & \beta_n \\ \beta_3 & a_{33} & a_{34} & \cdots & a_{3n} \\ \beta_4 & a_{34} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \beta_n & a_{3n} & \cdot & \cdot & \cdot \end{vmatrix},$$

which is symmetric and of order $n - 1$: for, any minor of A_1 which does not contain any element of the second row or column is also a minor of A_2 ; any minor which contains elements of both the first and the second row (column) is zero; and one which contains the elements of the second but not of the first row (column), and does not come under the case just mentioned, is i times the minor obtained by replacing the second row (column) by the first.

Since principal minors of A_2 are also principal minors of A_1 and the order of A_2 is one less than the order of A_1 while the ranks are the same, the fact that one principal minor of order r does not vanish follows readily by induction.

The proof may also be made by transforming A_1 by the non-singular matrix

$$C \equiv \begin{vmatrix} 1 & 0 & & & \\ i & 1 & & & \\ & & 1 & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 1 \end{vmatrix},$$

which gives

$$C'A_1C = \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & -\alpha & i\beta_1 & \cdots & i\beta_n \\ 0 & i\beta_1 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & i\beta_n & a_{3n} & \cdots & a_{nn} \end{vmatrix}.$$

In conclusion we may notice that the characteristic equation of A_1 is the same as that of the matrix obtained by replacing the first and second rows of A_1 by zeros.

ERRATA

Volume 15.

On the Projective Differential Geometry of Plane Anharmonic Curves.
By SAMUEL W. REAVES. Professor A. M. Harding has called my attention to the fact that equations (33) and (34) should read

$$(33) \quad 2e^{2t}x^2 + 8e^txy - y^2 + (10 - 4t)e^{2t}x - (2 + t)8e^ty + (2t^2 - 10t + 17)e^{2t} = 0,$$

$$(34) \quad y = -2e^{x-3/2}.$$

S. W. R.